Occurrence of Diffusion Equation

1-D Heat Equation

In a metal rod with non-uniform temperature, thermal energy (i.e. Heat) will flow from the high temperature region to the relatively low temperature region by the process of Diffusion. To formulate the heat flow problem, we need the following three physical principles:

- Heat energy (E) of an uniform body

$$E = cmT$$

Where c is the specific heat of the body, m is the mass of the body and T is the temperature.

- Fourier's Law of Heat transfer:

$$\frac{Rate\ of\ heat\ transfer}{Area} = -K_0 \frac{\partial T}{\partial x}$$

Where K_0 is the thermal conductivity. In other words, heat is transferred from the high temperature region to the low temperature region.

- Conservation of Energy.

Now, consider a uniform rod of length l with an uniform temperature lying on the x-axis from $x = 0$ to $x = l$. Uniformity means specific heat, density, conductivity and cross section area are constant.

Let an heat source is placed on the one end of the rod, and consider an arbitrary thin slice of the rod of width Δx between x and $\mathrm{x} + \Delta x$. The slice is so thin that the temperature throughout the slice is $T(x, t)$. Thus,

$$Heat\ Energy\ of\ Segment\ \Delta x = c \times \rho A \Delta x \times T(x,t) = c\rho A \Delta x T(x,t)$$

Where ρ is the density of the rod and A is the area of cross-section.

Now, by conservation of energy, we have

$$Change\ of\ heat\ energy\ of\ the\ time\ segement\ \Delta \mathrm{t} = \text{heat in from left boundary} - \text{heat out from right boundary}$$

Or

$$c\rho A \Delta x T(x, t+\Delta t) - c\rho A \Delta x T(x,t) = \Delta t A \left(-K_0 \frac{\partial T}{\partial x}\right)_x - \Delta t A \left(-K_0 \frac{\partial T}{\partial x}\right)_{x+\Delta x}$$

Or

$$\frac{T(x, t+\Delta t) - T(x,t)}{\Delta t} = \frac{K_0}{c\rho}\left(\frac{\left(\frac{\partial T}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial T}{\partial x}\right)_x}{\Delta x}\right)$$

On taking limits $\Delta x \to 0, \Delta t \to 0$, we get

$$\frac{\partial T}{\partial t} = \frac{K_0}{c\rho} \frac{\partial^2 T}{\partial x^2}$$

- The above partial differential equation is called 1-D Heat equation. This equation is applicable on whole rod as the slice was chosen arbitrarily.

Initial and the Boundary Conditions for 1-D heat equation

- Initial condition (IC): the initial temperature distribution in the rod $T(x, 0)$.
- Boundary Condition (BC): Specifying the temperature at the end of the rod. For instance, standard BCs for the end $x = 0$ can be given by following expression:

I. Temperature Prescribed at a boundary. For $t > 0, T(0, t) = T_1(t)$.

II. Insulated boundary. The heat flow can be prescribed at the boundaries, $-\mathrm{K}_0 \frac{\partial T}{\partial x}(0, t) = \phi_1(t)$.

III. Mixed conditions. An equation involving $T(0, t), \frac{\partial T}{\partial x}(0, t), etc.$

Example

Consider a rod of length l with insulated sides is given an initial temperature distribution of $f(x)$ degree C, for $0 < x < l$. Find $T(x,t)$ at subsequent times $t > 0$ if end of rod are kept at 0^o C.

Mathematical Formulation of the problem:

PDE:

$$\frac{\partial T}{\partial t} = \frac{K_0}{c\rho}\frac{\partial^2 T}{\partial x^2}, \qquad for\ \ 0 < x < l$$

IC:

$$T(x,0) = f(x), \qquad for\ \ 0 < x < l$$

BC:

$$T(0,t) = T(l,t) = 0, \qquad for\ \ t > 0$$

Extension to Higher Dimensions: 3D Heat Equation

- Let V be an arbitrary 3D-domain bounded by a closed surface S and let $\bar{V} = V \cup S$, that is domain V including the enclosing surface S. Let $T(x, y, z, t)$ be the temperature at any point (x, y, z) at time t in $\bar{V}$.
- Heat flow is from high heat region to relatively low heat region. In this case, it is governed by the following 3D-Fourier's law

$$q(r, t) = -K\nabla T(r, t)$$

Where $q(r, t)$ is the heat flux at time t at point (x, y, z) represented by the position vector r, and K is the thermal conductivity.

- If $\hat{n}$ be the outward normal vector to an infinitesimally small element dS of the enclosing surface S, then the heat flowing out through the elemental surface dS in unit time is given by

$$dQ = (q.\hat{n})dS$$

- The amount of heat dQ needed to raise the temperature of the elemental mass $dm = \rho dV$ to the value T is given by $dQ = C\rho T dV$, where C is the specific heat of the solid. Therefore, on integrating over volume, we get

$$Q = \iiint_V C\rho T dV$$

Or,

$$\frac{dQ}{dt} = \iiint\limits_V C\rho \frac{\partial T}{\partial t} dV$$

- Now, on balancing in and out heat: The rate of energy storage in V is equal to the sum of the rate of heat entering V through its bounding surface and the heat produced within V. We get

$$\iiint\limits_V C\rho \frac{\partial T}{\partial t}(r,t) dV = -\iint\limits_S q.\hat{n} dS + \iiint\limits_V H(r,T,t) dV$$

Where $H(r,T,t)$ is the heat generated within solid following mechanical and chemical reactions.

- By the divergence theorem, we can write

$$\iiint\limits_V \left(C\rho \frac{\partial T}{\partial t}(r,t) + \nabla.q(r,t) - H(r,T,t) \right) dV = 0$$

Or,

$$C\rho \frac{\partial T}{\partial t}(r,t) = -\nabla.q(r,t) + H(r,T,t) = -\nabla.\big(K\nabla T(r,t)\big) + H(r,T,t)$$

- Further simplification leads to

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}(r,t) = \nabla^2 T(r,t) + \frac{1}{K}H(r,T,t)$$

Where $\alpha = K/\rho C$ is the thermal diffusivity of the medium. Notice that in the absence of mechanical or chemical heat generation within the solid, we have $H(r,T,t) = 0$, therefore in this case we get following 3D-heat equation:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}(r,t) = \nabla^2 T(r,t)$$

Boundary Conditions (BCs)

I. Boundary Condition-I: Temperature is prescribed all over the boundary surface : $T = G(r,t)$which is the profile of temperature over the boundary surface at time t this type of condition is called the Drichlet's boundary condition. Special case $\mathrm{T}(r,t) = 0$ is called the homogeneous boundary condition.

II. Boundary Condition-II: The flux of heat, i.e. the normal derivative of the temperature $\partial T/\partial n$is prescribed on the boundary surface:$\partial T/\partial n = f(r,t)$. This is called the Neumann condition. Again the special case $\partial T/\partial n = 0$, i.e. T is time dependent only, is called the homogenous boundary condition.

III. Boundary Condition-III: It is a mixed type condition. A linear combination of the temperature and its normal derivative is prescribed on the boundaries, i.e.,

$$K\frac{\partial T}{\partial n} + hT = G(r,t)$$

Where h and K are constants. This type of boundary condition is called the Robin's condition.

Again, we can write the homogeneous boundary conditions:

$$K\frac{\partial T}{\partial n} + hT = 0$$

as a special case.